The Nyquist Stability Criterion

- Born in 1889 in Sweden
- Died in 1976, USA
- Yale PhD, 1917
- Career at Bell Labs
- 138 patents
- Nyquist diagram, criterion, sampling theorem
- Laid the foundation for information theory, data transmission and negative feedback theory

Frequency Response Plots

$$G(j\omega) = G(s)\Big|_{s=j\omega}$$
$$= |G(j\omega)|\, e^{j\phi(\omega)}$$
$$= |G(j\omega)| \angle \phi(\omega)$$
$$= R(\omega) + jX(\omega)$$

$$|G(\omega)| = \sqrt{\mathrm{Re}^2(\omega) + \mathrm{Im}^2(\omega)}$$

$$\phi = \tan^{-1} \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}$$

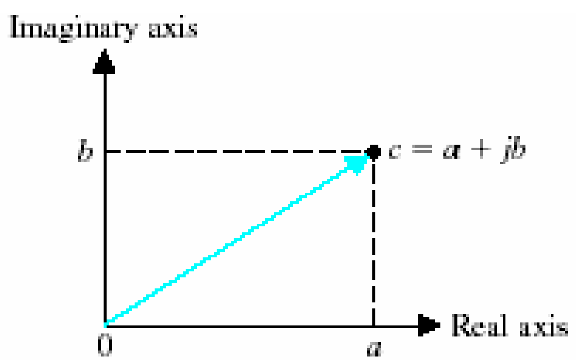


FIGURE G.1 Rectangular form of a complex number.

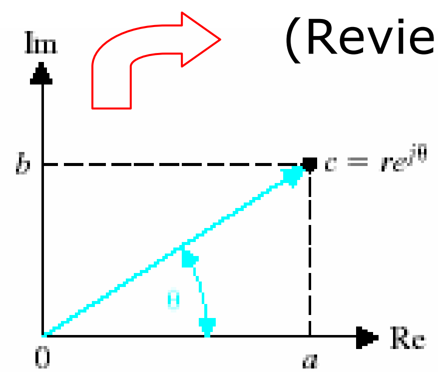
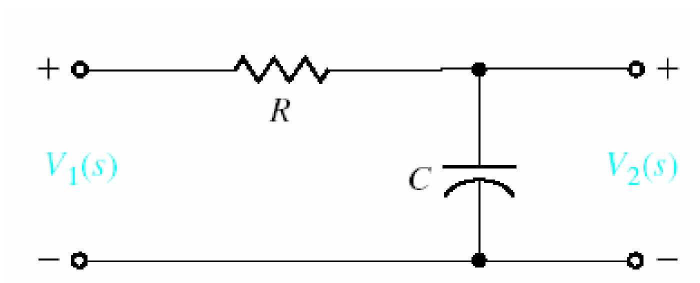


FIGURE G.2 Exponential form of a complex number.

(Review Appendix G in textbook)

Frequency Response Plots



$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs+1}$$

1st Order system

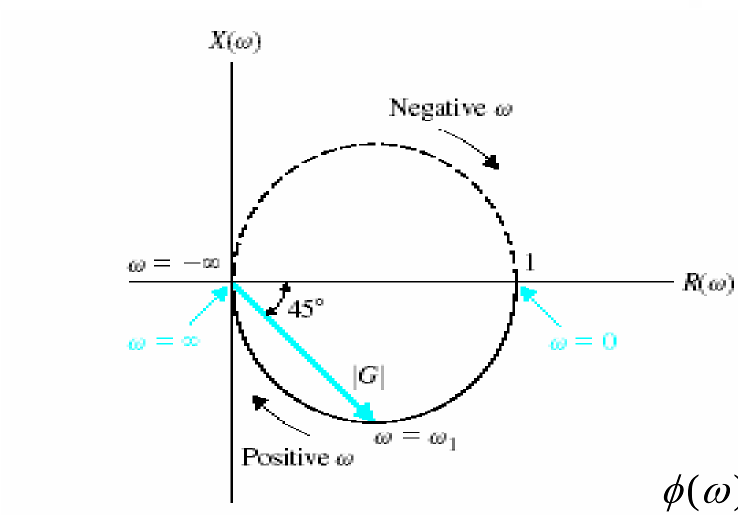
$$G(j\omega) = \frac{1}{j\omega RC+1} = \frac{1}{j\omega/\omega_1+1}$$

$$\omega_1 = 1/RC$$

$$G(j\omega) = R(\omega) + jX(\omega)$$

$$= \frac{1 - j\omega/\omega_1}{(\omega/\omega_1)^2+1}$$

$$= \frac{1}{(\omega/\omega_1)^2+1} - \frac{j\omega/\omega_1}{(\omega/\omega_1)^2+1}$$



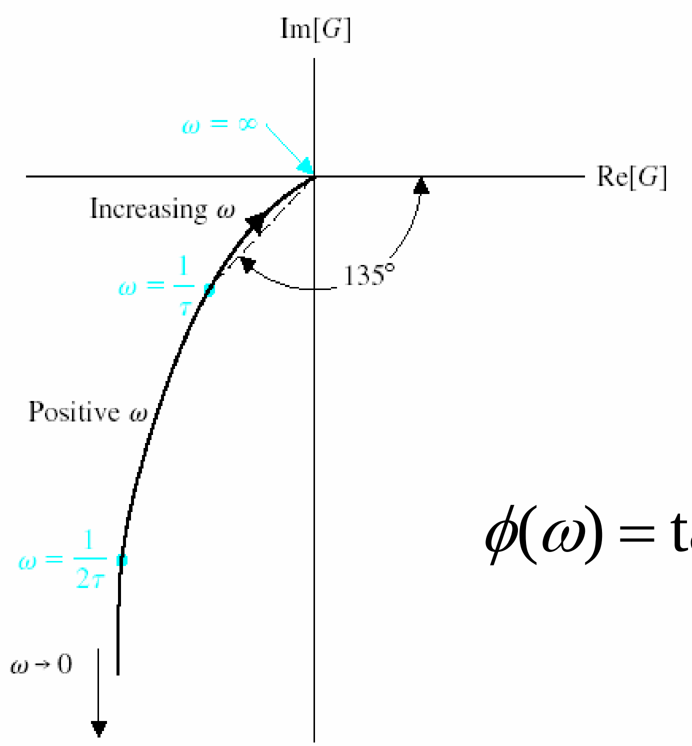
$$\phi(\omega) = \tan^{-1}\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)} = \tan^{-1}\frac{-\omega/\omega_1}{1} = -\tan^{-1}\left(\frac{\omega}{\omega_1}\right)$$

Polar Plot or Nyquist Diagram

$$G(s) = \frac{K}{s(s\tau + 1)}$$ 2rd system $$G(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)} = \frac{K}{j\omega - \omega^2\tau}$$

$$= \frac{-K\omega^2\tau}{\omega^2 + \omega^4\tau^2} - \frac{jK\omega}{\omega^2 + \omega^4\tau^2}$$

$$|G(\omega)| = \frac{K}{(\omega^2 + \omega^4\tau^2)^{\frac{1}{2}}}$$



$$\phi(\omega) = \tan^{-1}\frac{\text{Im}(\omega)}{\text{Re}(\omega)} = \tan^{-1}\frac{-K\omega}{-K\omega^2\tau} = \tan^{-1}\left(\frac{1}{\omega\tau}\right)$$

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Polar Plots and the Nyquist Criterion

- We have seen how to generate polar plots of the frequency response.
- The Nyquist criterion is a way of interpreting the polar plot to determine stability
- The frequency response in the entire right-half *s*- plane with $s=-j\infty$ to $s=+j\infty$ is the critical section of the Nyquist contour.

Mapping Contours in the s-Plane

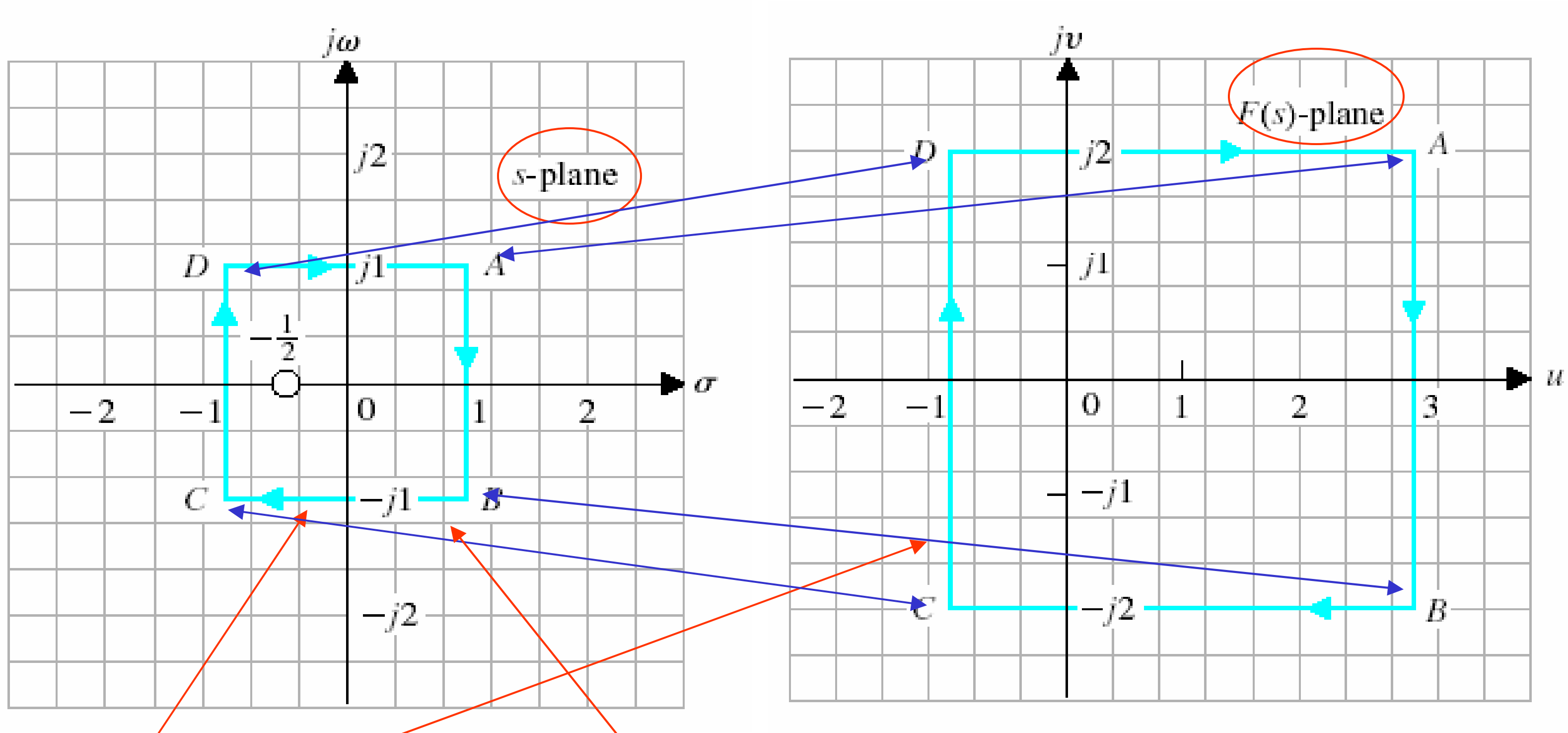
- Nyquist stability criterion based on Cauchy theorem on functions of a complex variable

- Mapping contours

$$1+L(s)=0$$

in the s-plane.

Mapping Contours in the s-Plane



Closed contours

$$F(s) = 2s + 1$$

Conformal mapping (angles are preserved)

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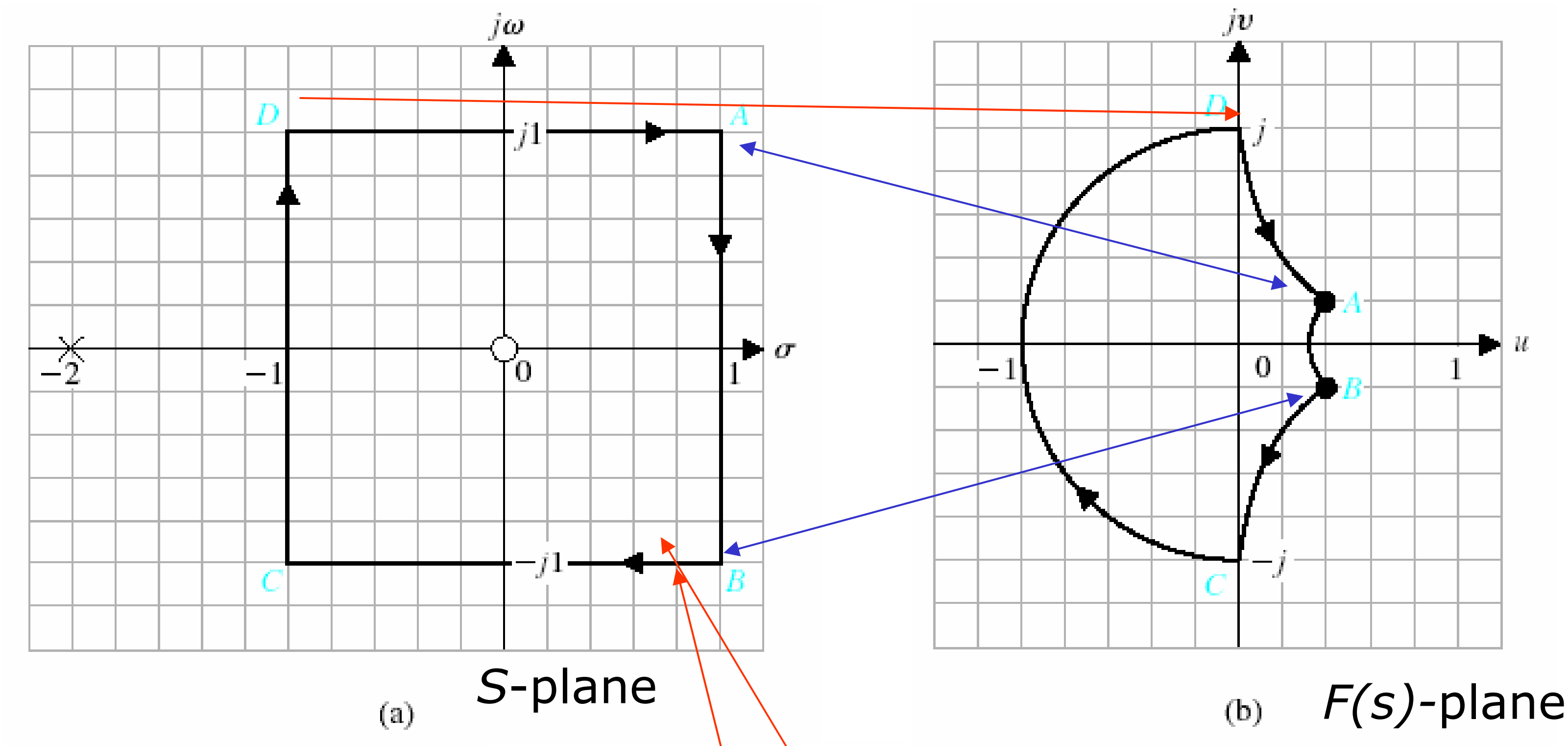
Mapping Contours in the s-Plane

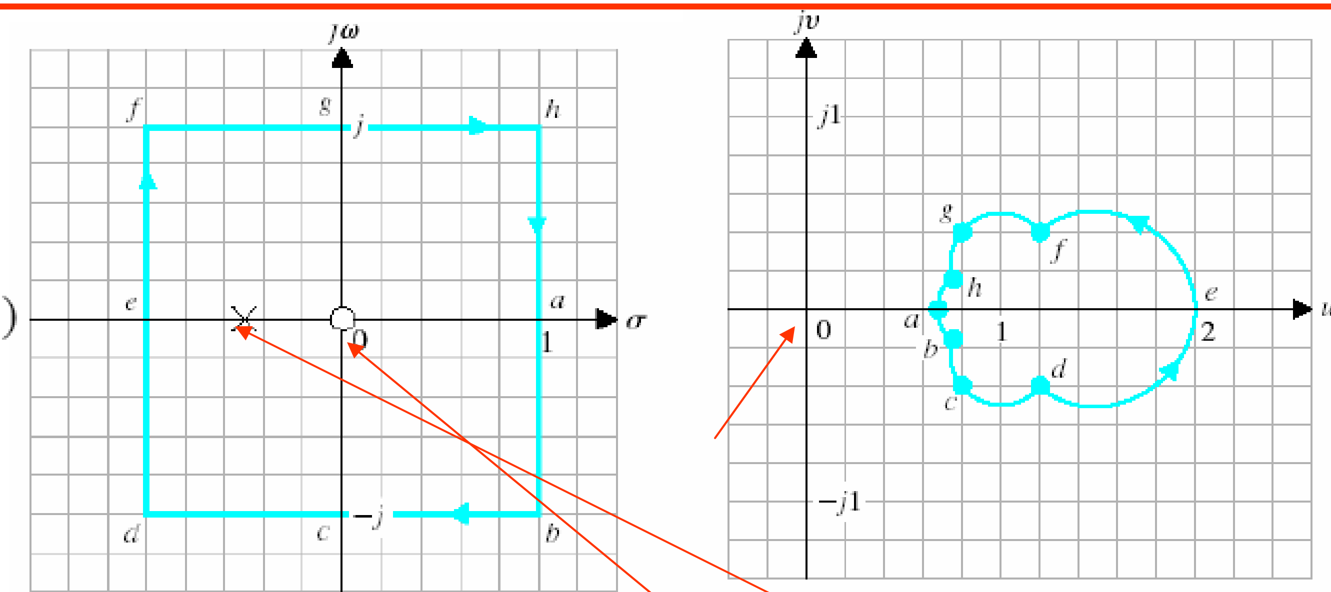


$$F(s) = \frac{s}{s+2}$$

Cauchy's Theorem: Principle of the Argument

If a contour Γ_s in the *s*-plane encircles ***Z*** zeros and ***P*** poles of ***F***(*s*) and does not pass through any poles or zeros of ***F***(*s*) and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the *F*(*s*)-plane encircles the origin of the *F*(*s*)-plane *N*=*Z*-*P* times in the ***clockwise direction***

2)



S-plane

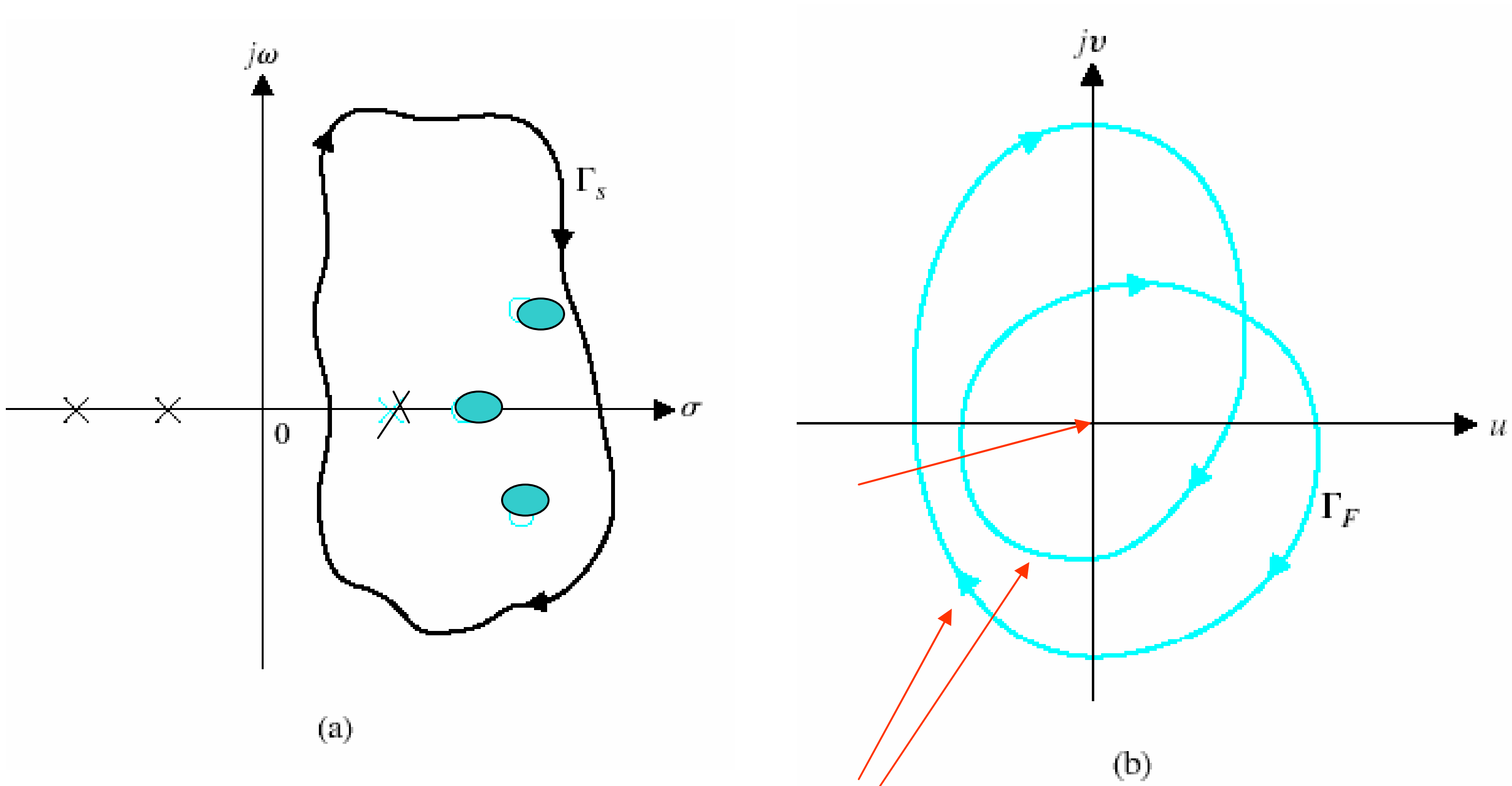
$F(s) = s/(s+1/2)$

F(s)-plane

?

N=1-1=0

Illustration of Cauchy's Theorem



$N=Z-P=3-1=2$

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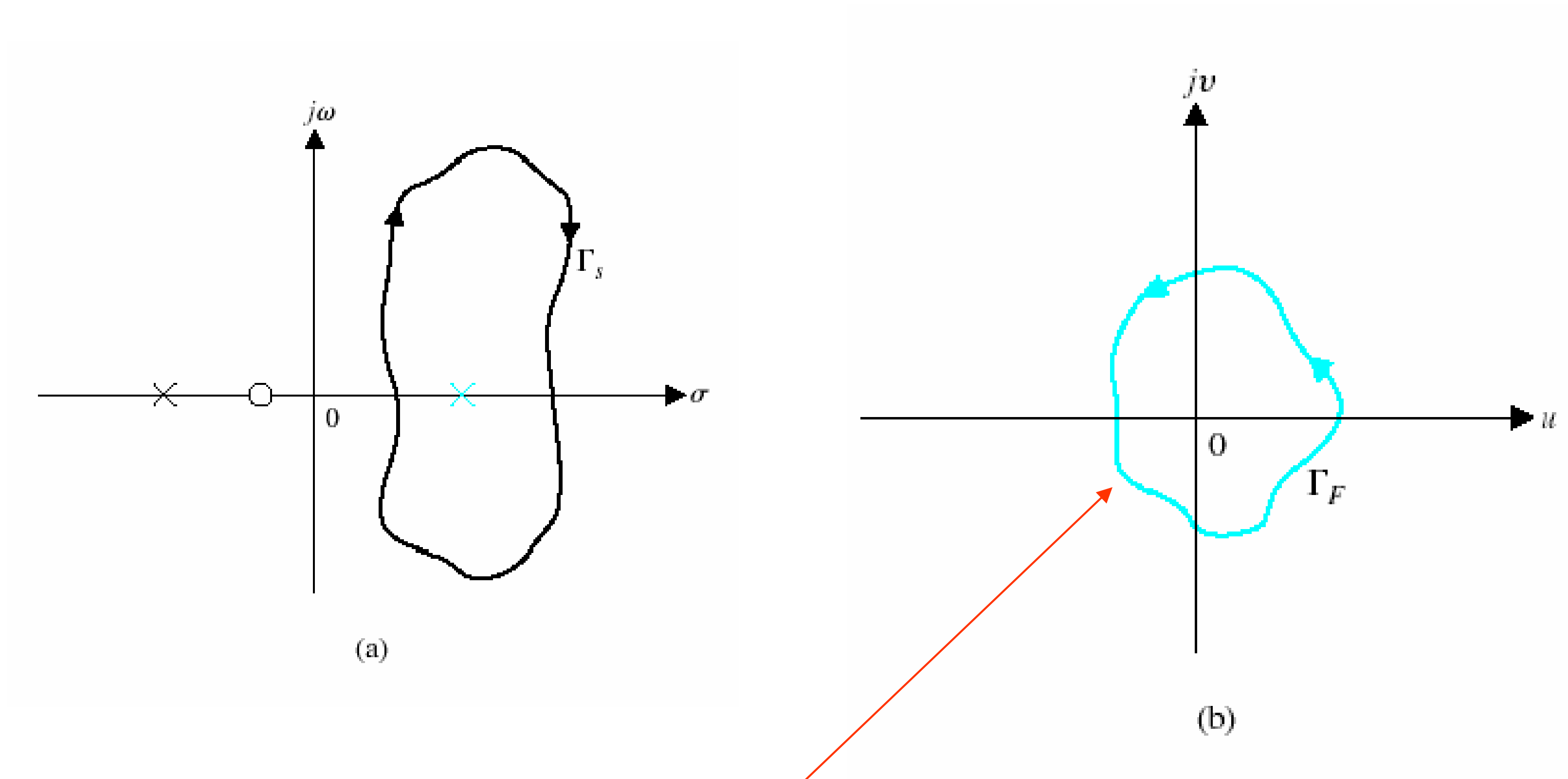
Illustration of Cauchy's Theorem

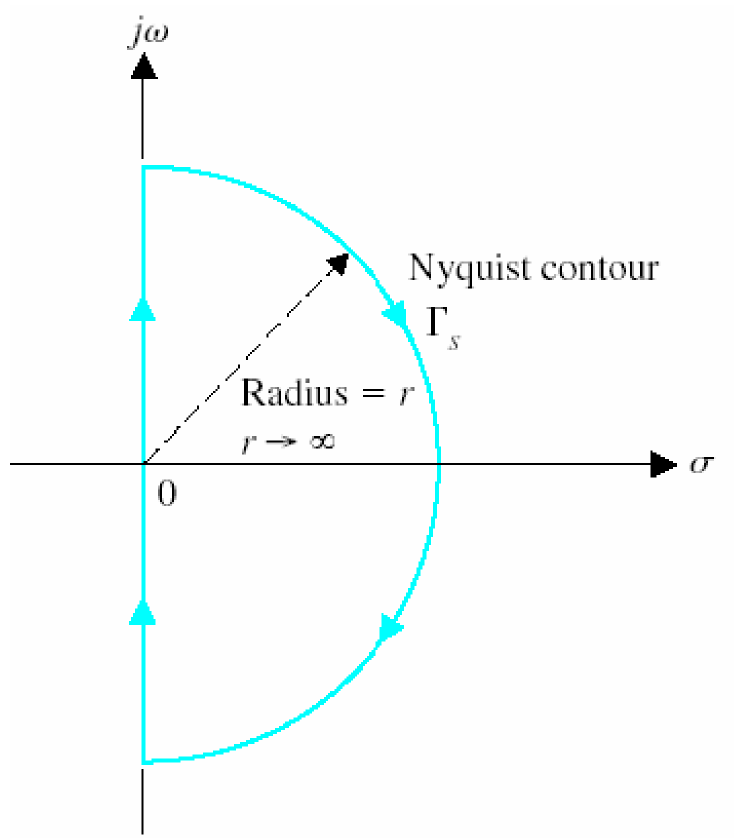


$N = Z - P = 0 - 1 = -1$

The Nyquist Criterion

The Nyquist contour encircles the entire right-half *s*- plane

$$F(s) = 1 + L(s) = \frac{K\prod_{i=1}^{n}(s+s_i)}{\prod_{k=1}^{m}(s+s_k)}$$



For the system to be stable, all zeros (or roots) of *F*(*s*) must lie in the left-hand s-plane.

$N=Z-P$

The number of unstable zeros of *F*(*s*) is thus

$Z=N+P$

Where *N* is the number of encirclements of the origin.

When the system is open- loop stable P=0 and then $Z=N$

The Nyquist Criterion

In practice it is more convenient to consider the mapping

$$L(s) = F(s) - 1$$

in which case becomes the number of clockwise encirclements of the -1 point.

The Nyquist contour encircles the entire right-half *s*- plane

Z is unstable zero and P is unstable pole if they are lied within Nyquist contour.

The Nyquist Criterion

$Z=N+P$ For stable system, the closed loop Z=0.

1) Open-loop stable plant $L(s)\ (P=0)$ Z=0 => N=0

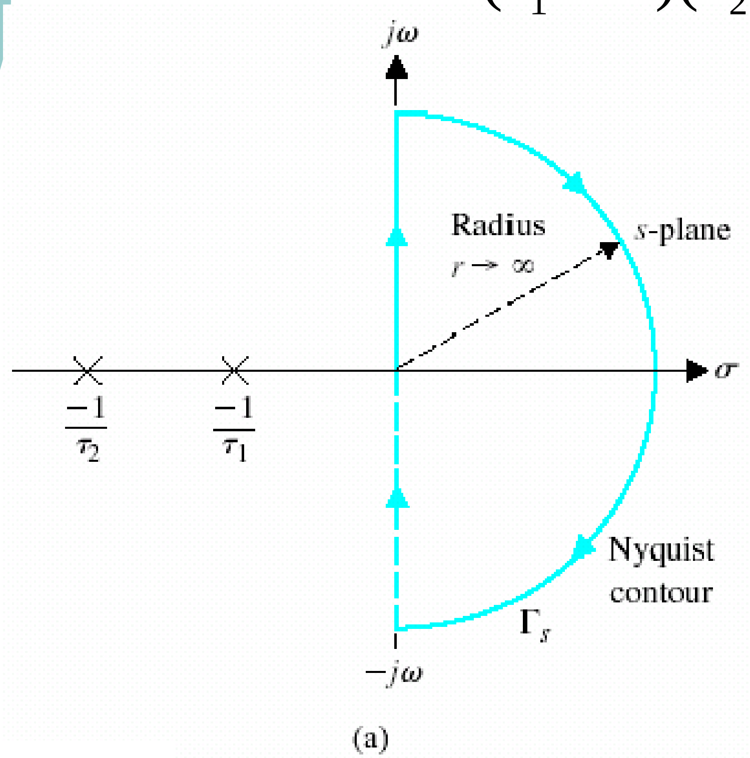
A feedback system is stable if and only if the contour Γ_L in the $L(s)$-plane does not encircle the (-1,0) point.

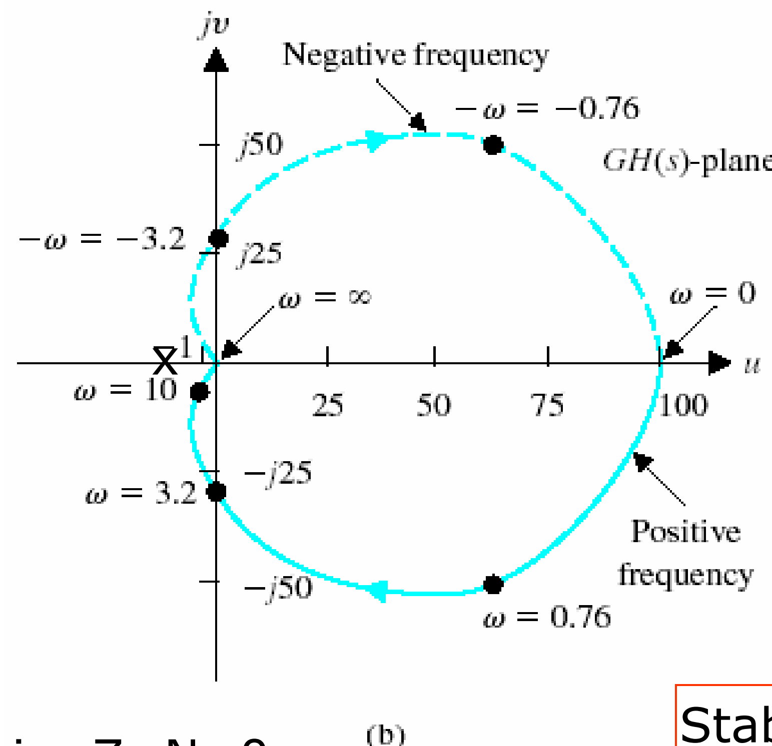
2) Open-loop unstable plant $L(s)\ (P\neq 0)$ Z=0 => N=-P

A feedback system is stable if and only if, for the contour Γ_L in the $L(s)$-plane, the number of counterclockwise of the (-1,0) point is equal to the number P of poles of $L(s)$ with positive real parts.

System with Two Real Poles

$$GH(s) = \frac{K}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{100}{(s+1)(0.1s+1)}$$





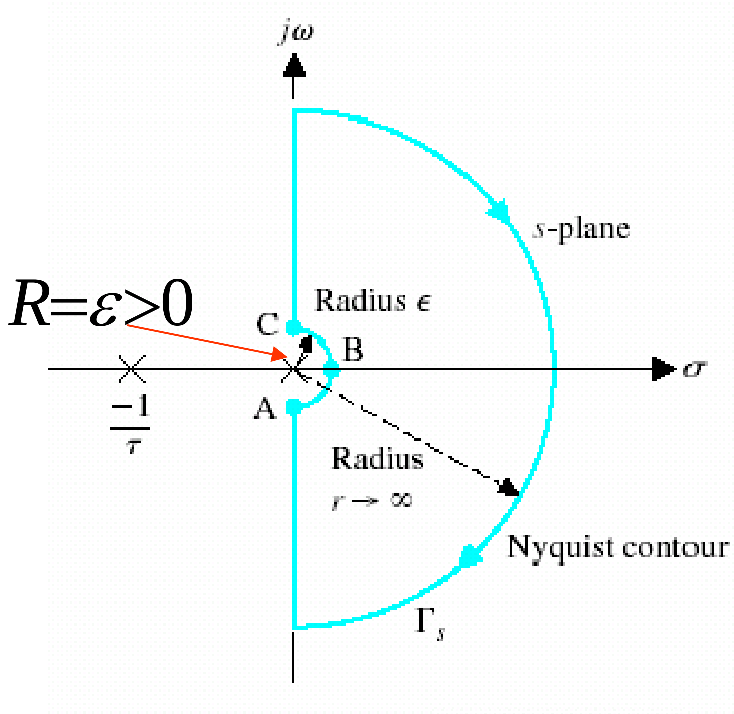
Stable

$P = 0$, hence for stability we require Z=N=0,

i.e. the contour must not encircle the -1 point in the *GH*(s)-plane.

System with a Pole at the Origin

$$GH(s) = \frac{K}{s(\tau s + 1)}$$



(a)

The Cauchy theorem requires that the contour cannot pass through the pole → Modified or indented Nyquist contour

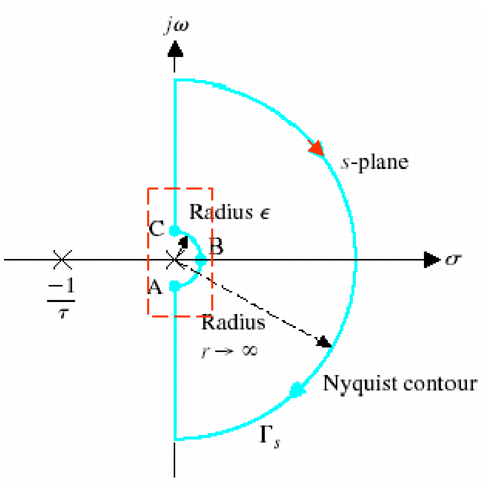
System with a Pole at the Origin

Small circular detour represented by $s = \varepsilon e^{j\phi}$ where

ϕ varies from -90° at $\omega = 0^-$ to $+90^\circ$ at $\omega = 0^+$.

As $\varepsilon \to 0$, $GH(s)$ is

$$\lim_{\varepsilon \to 0} GH(s) = \lim_{\varepsilon \to 0}\left(\frac{K}{\varepsilon e^{j\phi}}\right) = \lim_{\varepsilon \to 0}\left(\frac{K}{\varepsilon}\right) e^{-j\phi}$$



(a)

$GH(s)$ for this portion is an infinite semi-circle going from $+90^\circ$ at $\omega = 0^-$ to -90° at $\omega = 0^+$ and passing through 0° at $\omega = 0$.

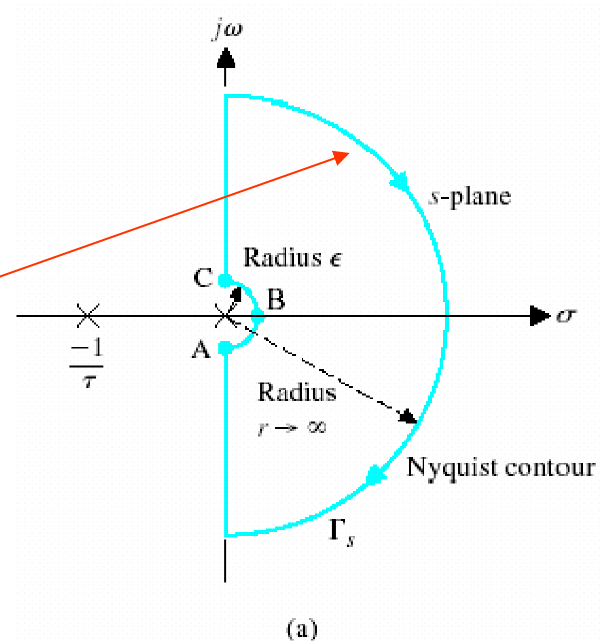
System with a Pole at the Origin

Portion from $\omega = +\infty$ to $\omega = -\infty$

$$\lim_{r\to\infty} GH(s) = \lim_{r\to\infty} \left| \frac{K}{(\tau r^2)} \right| e^{-2j\phi}$$

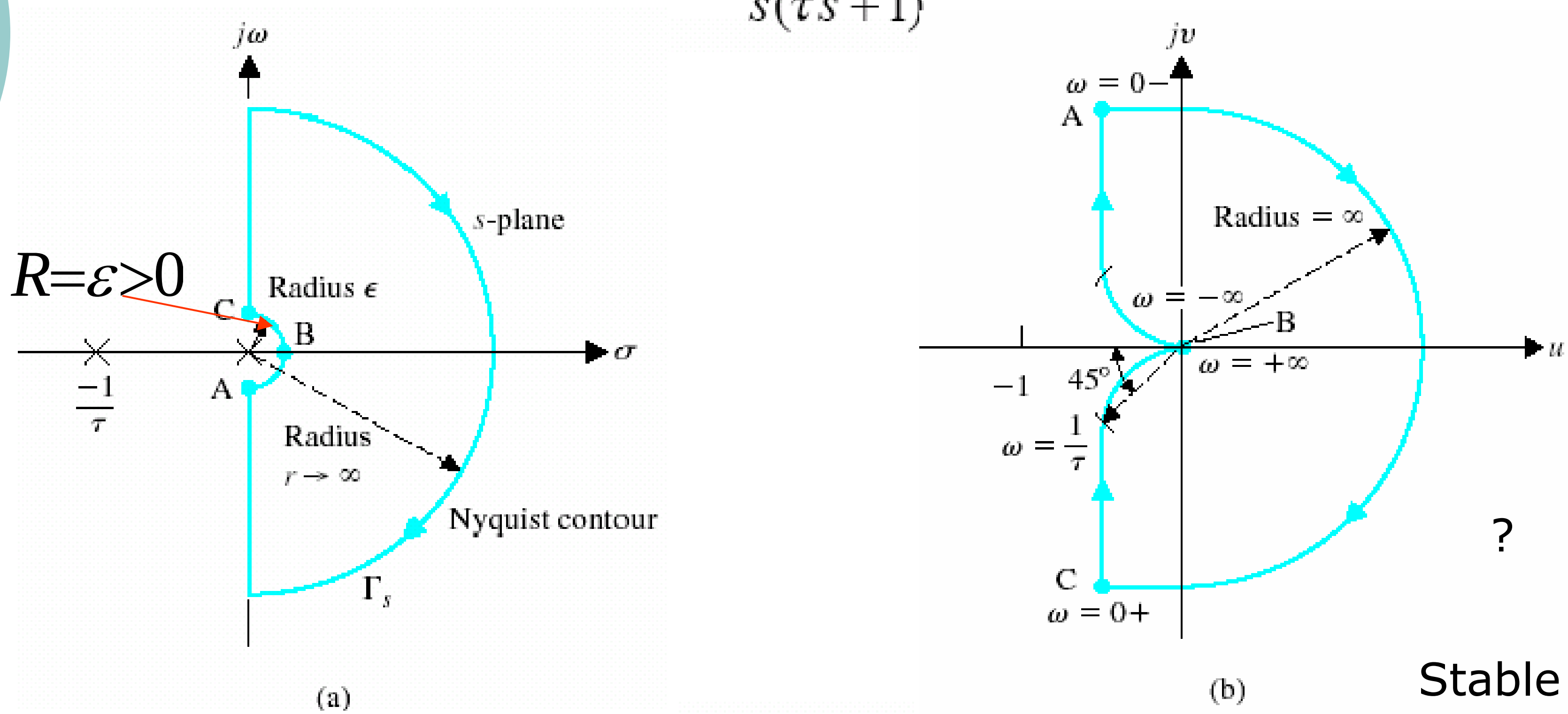
-180° at $\omega = +\infty$ to $+180^\circ$ at $\omega = -\infty$

Magnitude is zero when *r* is infinite



System with a Pole at the Origin

$$GH(s) = \frac{K}{s(\tau s + 1)}$$



The Cauchy theorem requires that the contour cannot pass through the pole → Modified or indented Nyquist contour

System with Three Poles

$$GH(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$GH(j\omega) = \frac{-K(\tau_1 + \tau_2) - jK(1/\omega)(1 - \omega^2 \tau_1 \tau_2)}{1 + \omega^2(\tau_1{}^2 + \tau_2{}^2) + \omega^4 \tau_1{}^2 \tau_2{}^2}$$

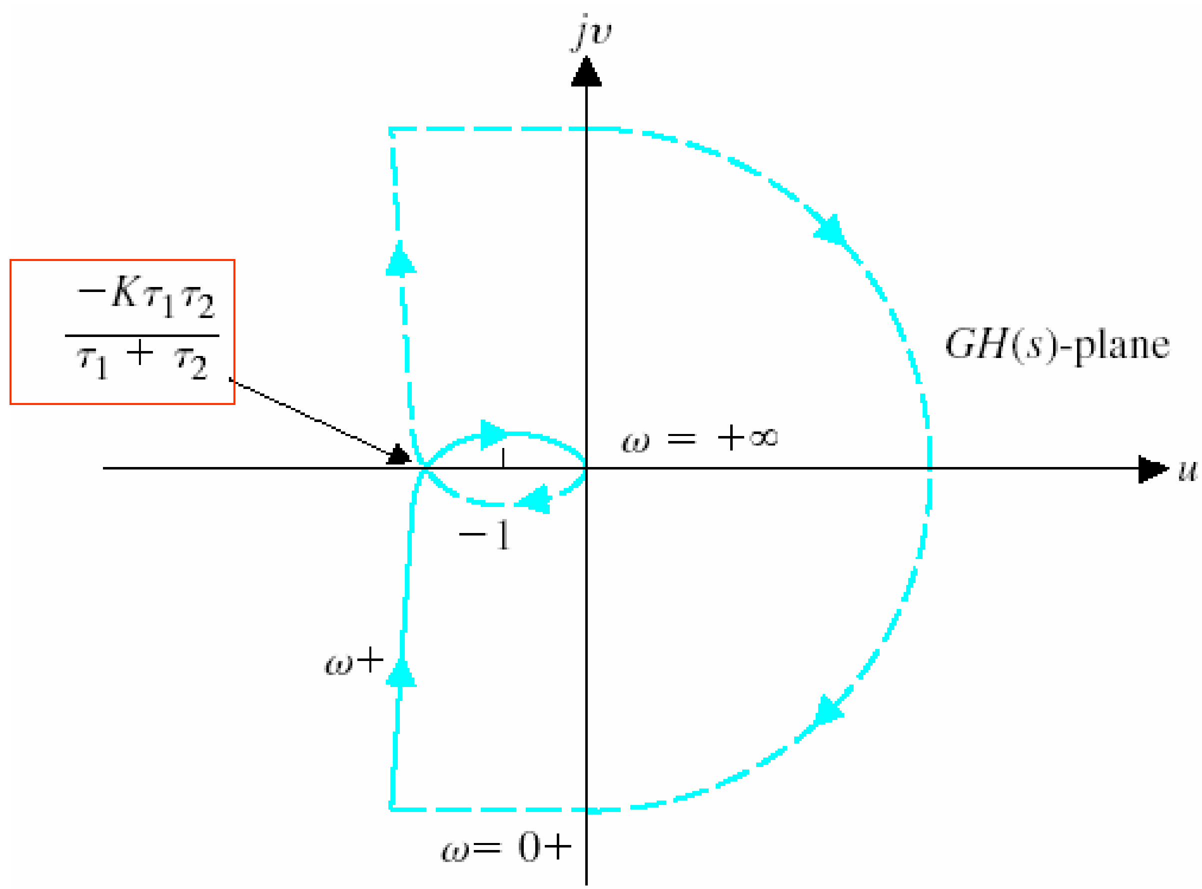
$$|GH(\omega)| = \frac{K}{\sqrt{\omega^4(\tau_1 + \tau_2)^2 + \omega^2(1 - \omega^2 \tau_1 \tau_2)^2}}$$

$$\phi(\omega) = -\tan^{-1}\omega\tau_1 - \tan^{-1}\omega\tau_2 - \pi/2$$

$$\lim_{\omega \to \infty} |GH(\omega)| = 0$$

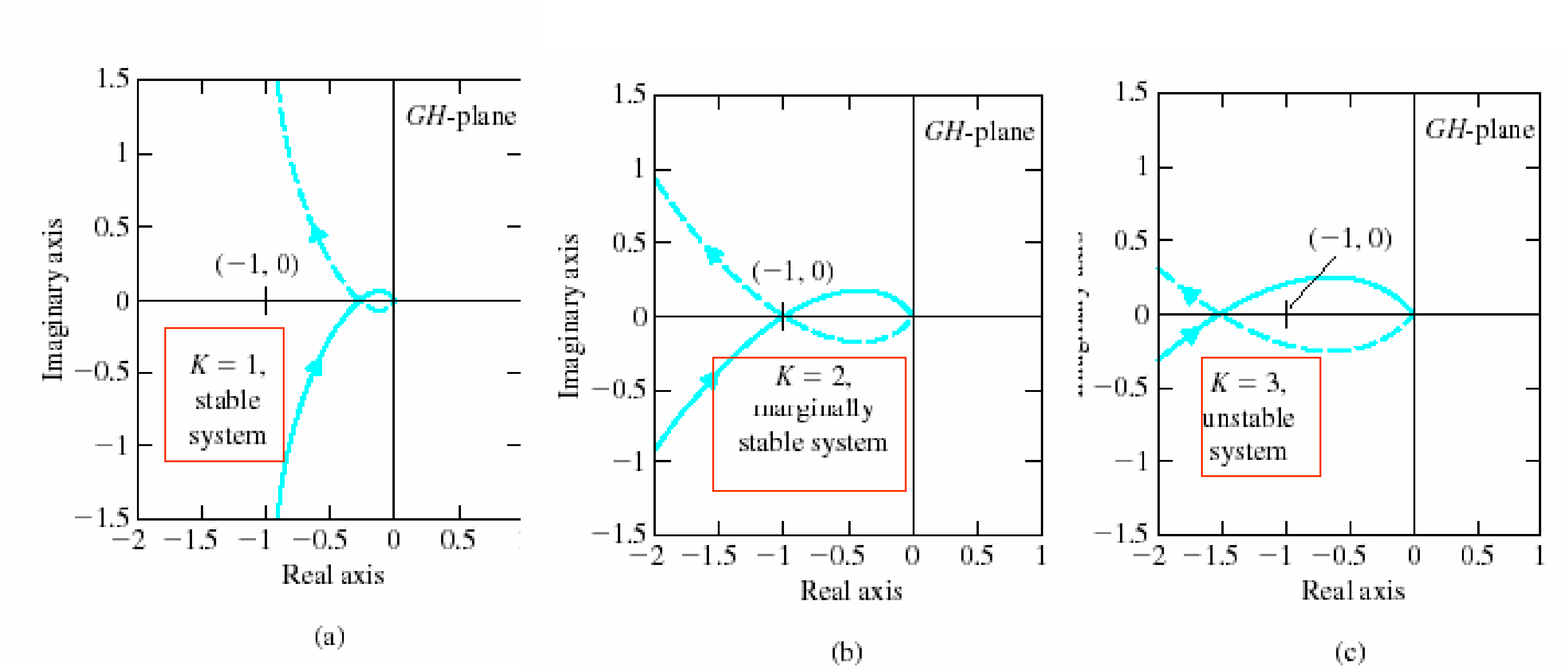
$$\lim_{\omega \to \infty} \phi(\omega) = -3\pi/2 = -270^\circ$$

System with Three Poles



The system is stable or not depending on the k value.

System with Three Poles



System with Two Poles at the Origin

$$GH(s) = \frac{K}{s^2(\tau s + 1)}$$

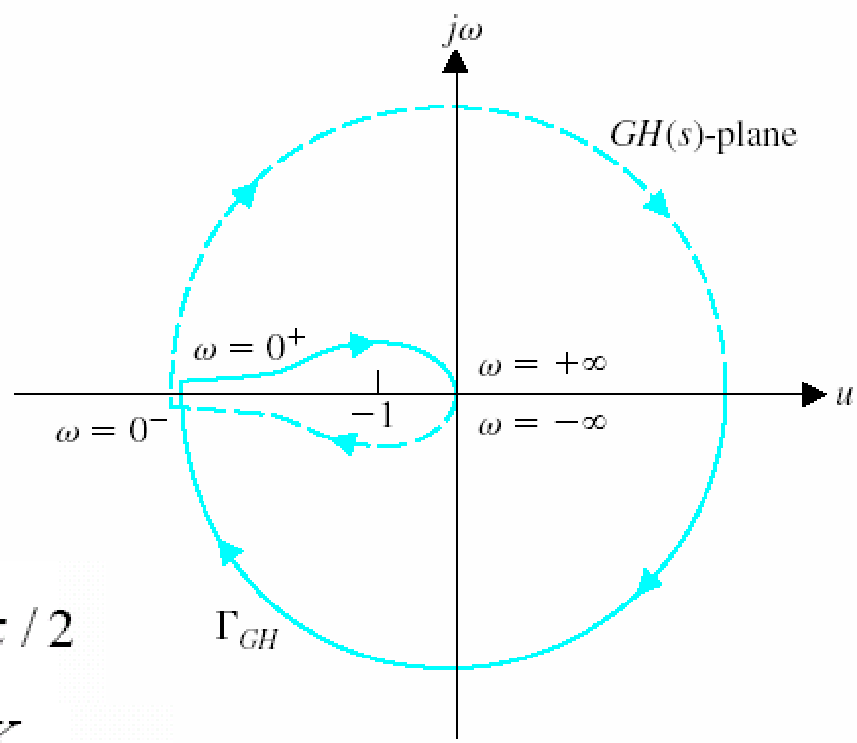
$$|GH(\omega)| = \frac{K}{\sqrt{\omega^4 + \tau^2\omega^6}}$$

$$\phi(\omega) = -\pi - \tan^{-1}\omega\tau$$

$$\lim_{\omega\to 0^+} GH(j\omega) = \left(\lim_{\omega\to 0^+}\frac{K}{\omega^2}\right)\angle -\pi$$

$$\lim_{\omega\to +\infty} GH(j\omega) = \left(\lim_{\omega\to +\infty}\frac{K}{\omega^3}\right)\angle -3\pi/2$$

With $s=\varepsilon e^{j\phi}$, $\lim_{\varepsilon\to 0} GH(s) = \lim_{\varepsilon\to 0}\frac{K}{\varepsilon^2}e^{-2j\phi}$



Angle of π at $\omega = 0^-$ to $-\pi$ at $\omega = 0^+$

N=2 hence *Z=2,* i.e. two roots in the RHP

Unstable for all K

System with a Pole in the RHP

$$GH(s) = \frac{K_1}{s(s-1)}$$

P=1, hence we require *N=-P=-1* for stability.

With $s = \varepsilon e^{j\phi}$ $-\pi/2 \le \phi \le \pi/2$

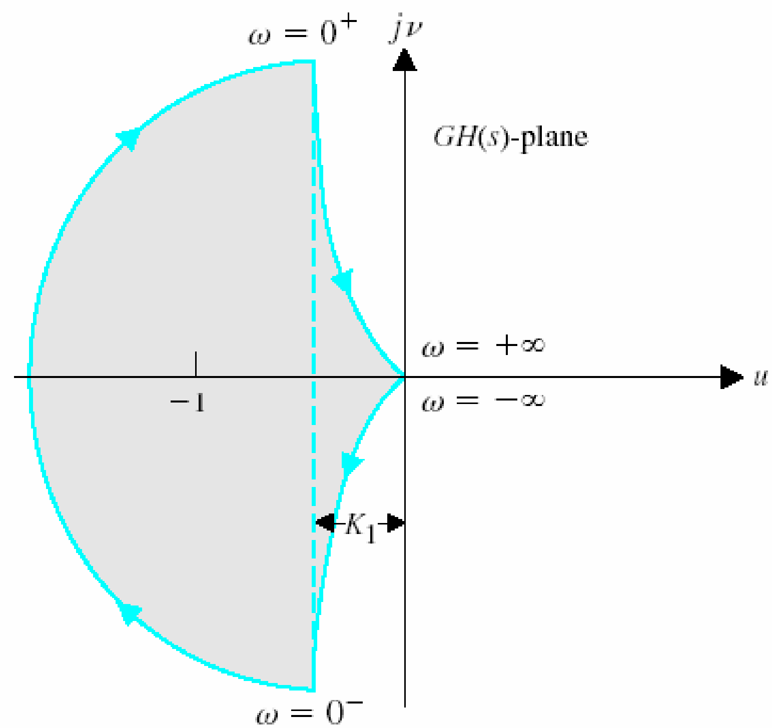
$$\lim_{\varepsilon \to 0} GH(s) = \frac{K_1}{-\varepsilon e^{j\phi}} = \left(\lim_{\varepsilon \to 0} \frac{K_1}{\varepsilon} \right) \angle -180^\circ - \phi$$

Infinite semicircle in the LHP

With $s = re^{j\phi}$

$$\lim_{r \to \infty} GH(s) = \lim_{r \to \infty} \left| \frac{K_1}{r^2} \right| e^{-2j\phi}$$

$$GH(j\omega) = \frac{K_1}{\sqrt{(\omega^2 + \omega^4)}} \angle \pi/2 + \tan^{-1}\omega$$



$N = 1$, hence $Z = N + P = 2$

Two roots in the RHP

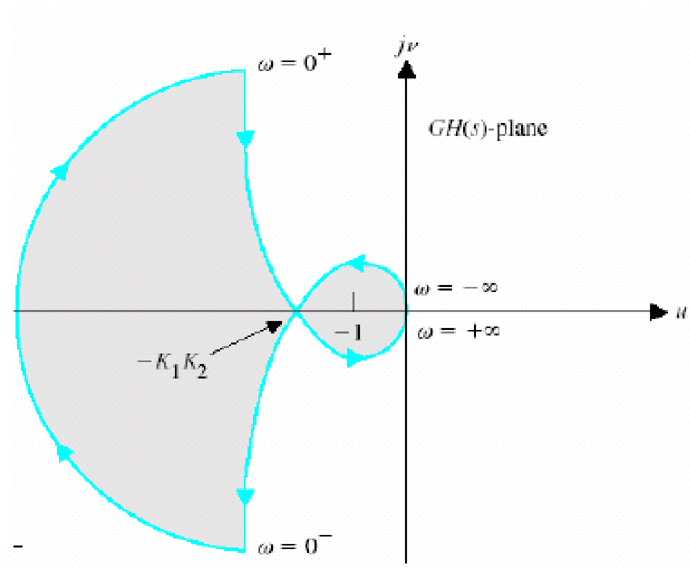
System with a Pole in the RHP

Adding derivative action

$$GH(s) = \frac{K_1(1+K_2 s)}{s(s-1)}$$

When $s = re^{j\phi}$

$$\lim_{r\to\infty} GH(s) = \lim_{r\to\infty} \left|\frac{K_1 K_2}{r}\right| e^{-j\phi}$$



$$GH(j\omega) = \frac{-K_1(\omega^2 + \omega^2 K_2) + j(\omega - K_2\omega^3)K_1}{\omega^2 + \omega^4}$$

Stable

Crosses real axis when $\omega - K_2\omega^3 = 0 \Rightarrow \omega^2 = 1/K_2$

$$GH(j\omega)\Big|_{\omega^2 = 1/K_2} = -K_1 K_2$$

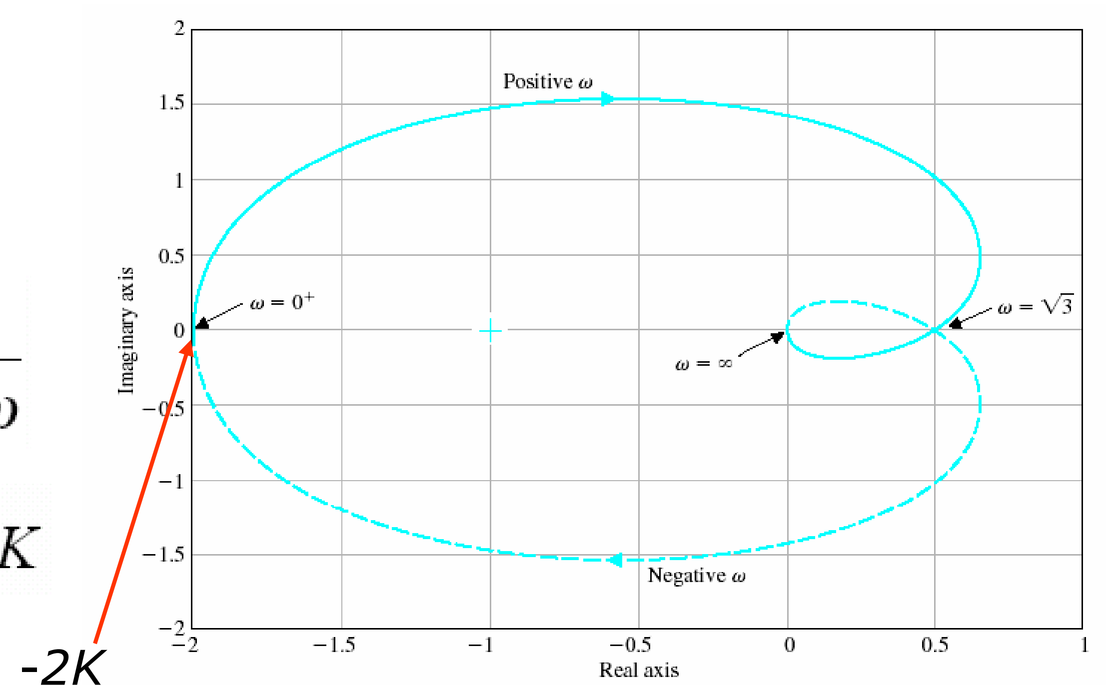
For $K_1 K_2 > 1$, $N = -1$ and $Z = N + P = -1 + 1 = 0$

System with a Zero in the RHP

$$GH(s) = \frac{K(s-2)}{(s+1)^2}$$

$$GH(j\omega) = \frac{K(j\omega - 2)}{(1-\omega^2) + 2j\omega}$$

$$\text{For } \omega = 0^+ \quad GH(j\omega) = -2K$$



$$N = 1, \text{ and } P = 0 \text{ mean } Z = N + P = 1$$

Unstable for $K > 1/2$

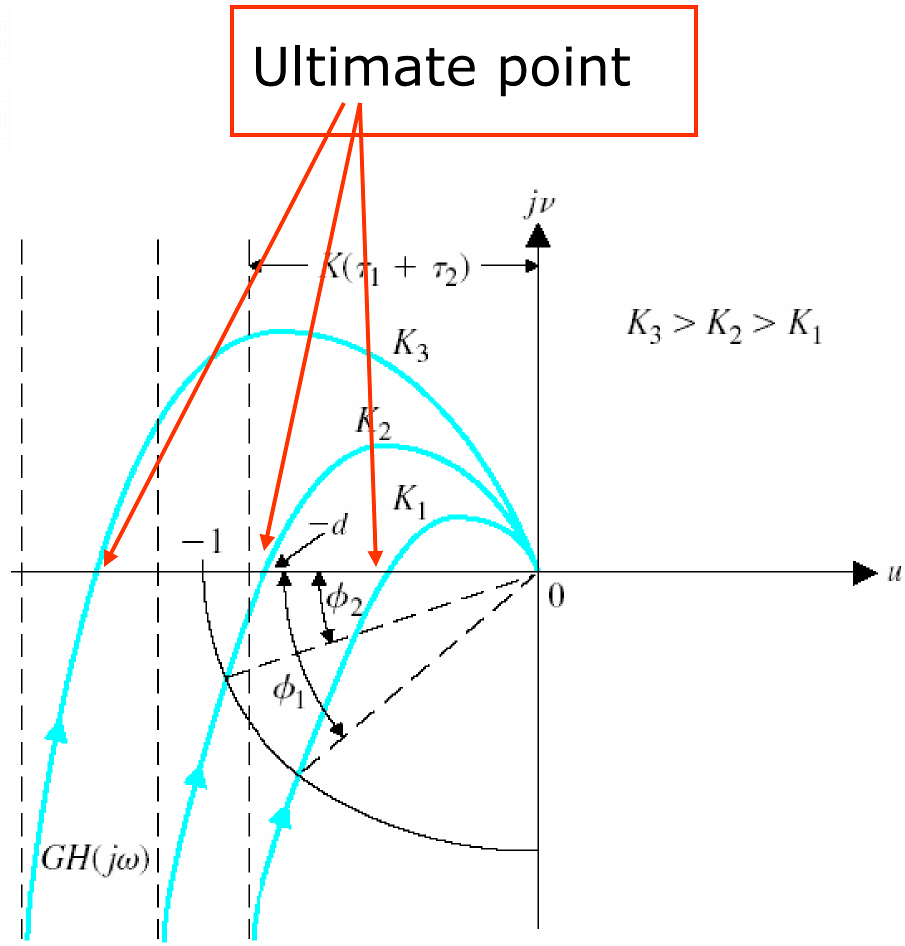
Relative Stability

$$GH(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

crosses the real axis at

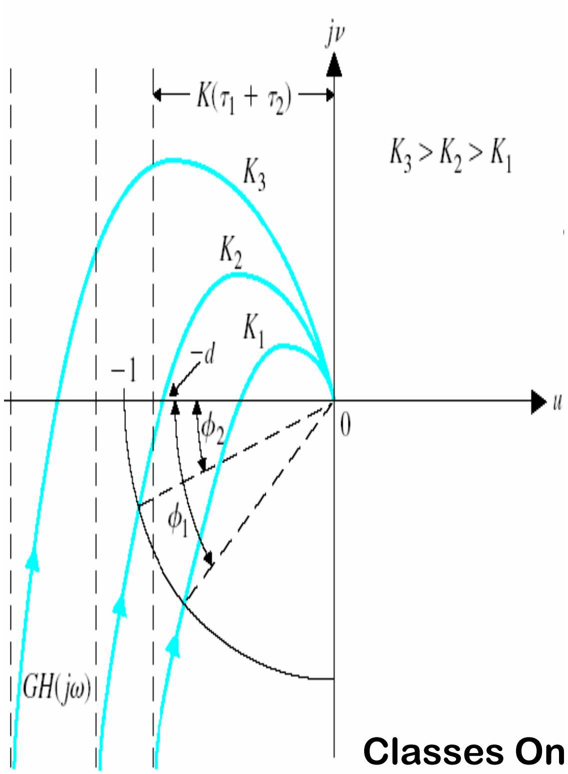
$$u = \frac{-K\tau_1\tau_2}{\tau_1 + \tau_2}$$

u=-1 when $K = \frac{\tau_1 + \tau_2}{\tau_1\tau_2}$



Relative Stability and Nyquist Criterion

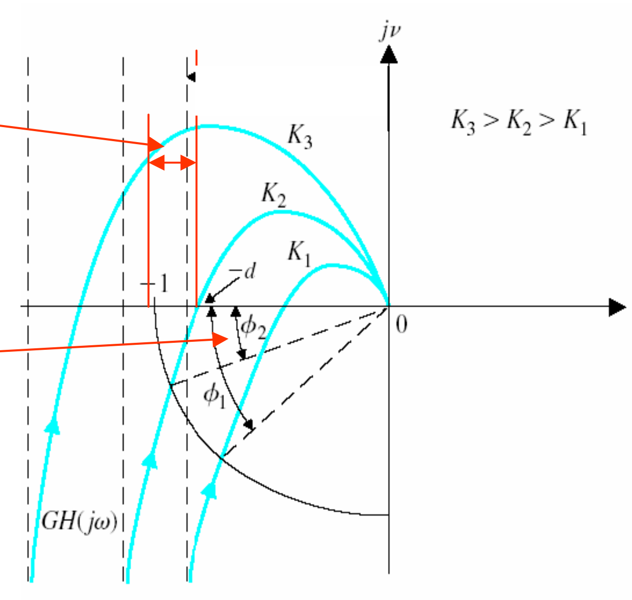
- Clearly, from the Nyquist criterion the distance between the (-1,0) point and the polar plot of the loop transfer function $GH(j\omega)$ is a measure of the relative stability of the system



- Note that the (-1,0) point on a Nyquist diagram corresponds to the 0 dB and -180 degrees on the Bode magnitude and phase diagrams

Relative Stability

- The **gain margin** is the increase in the system gain when phase =- 180^0 that will result in a marginally stable system with intersection of the -1+j0 on the Nyquist diagram



- The **phase margin** is the amount of phase shift of the GH Nyquist plot at unity magnitude that will result in a marginally stable system with intersection of the -1+j0 point on the Nyquist diagram

Gain Margin

The factor by which the gain at ultimate point can be increased to reach marginal stability

$$\frac{1}{d} = \frac{1}{|L(j\omega_{180})|}$$

Expressed in dB:

$$M_G = 20\log(\frac{1}{d}) = -20\log d$$

$$M_G = -20\log|L(j\omega_{180})| \text{ dB}$$

Reasonable values are gain margins in the range [2 5], or in dB [6 14]dB.

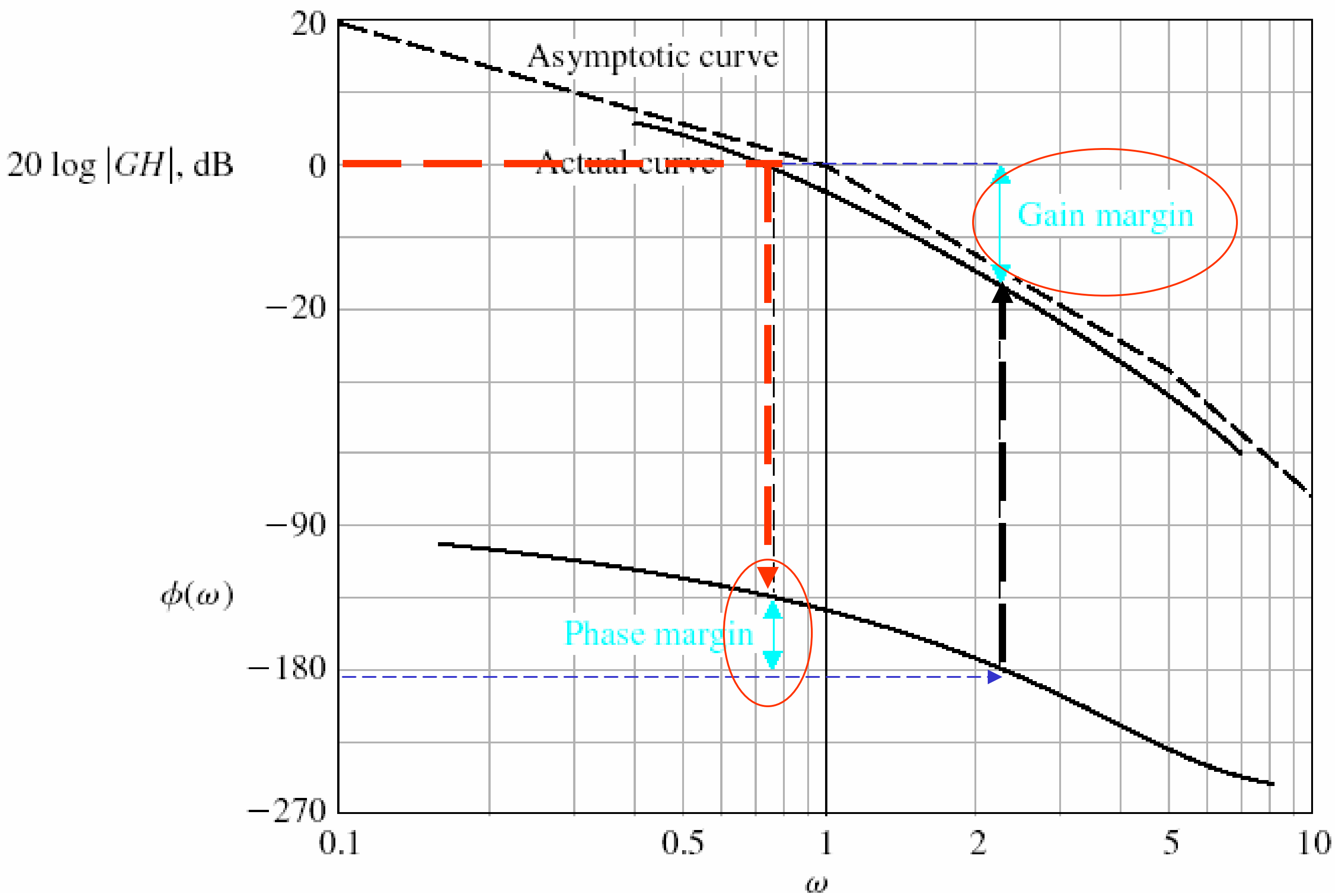
Phase Margin

The phase margin is $\phi,$ the difference between $-\pi$ and the phase of the loop at the crossover frequency ω_c

$$M_\phi = \pi + \arg L(j\omega_c)$$

Reasonable values are phase margins in the range [30 60] degrees

Gain and Phase Margins on the Bode Diagram



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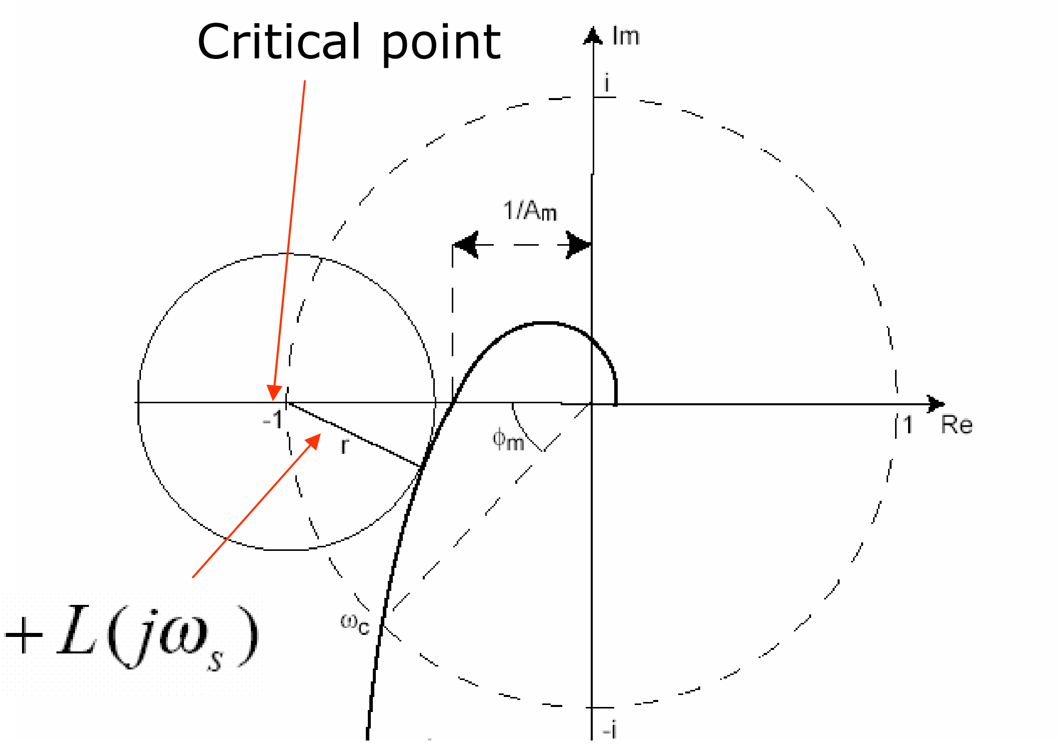
Sensitivity and Nyquist Plot

The radius r of the circle centered at -1 and tangen to *L* is the reciprocal of the nominal sensitivity peak

$$r = |1 + L(j\omega_s)| = |S(j\omega_s)|^{-1}$$



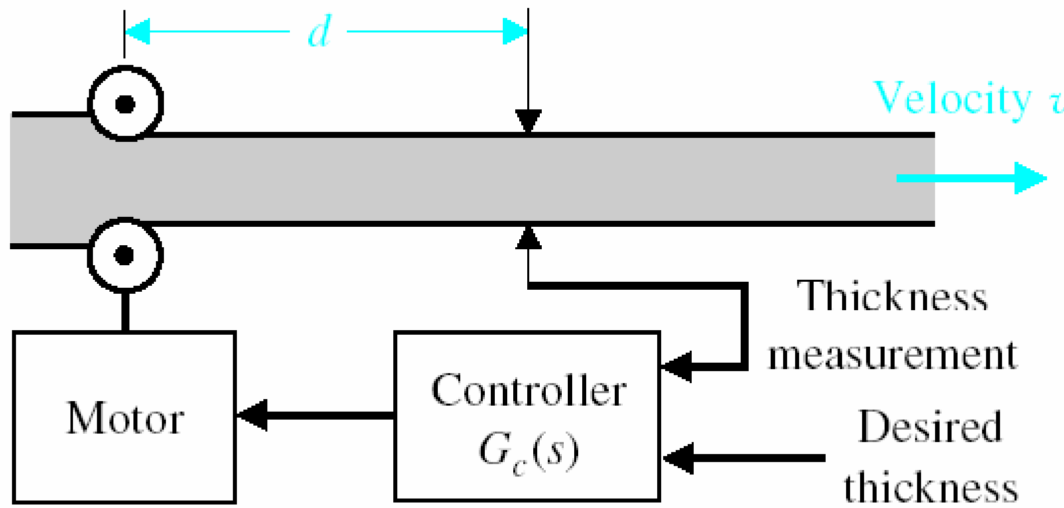
Effect of Time Delay

- The Nyquist criterion is applicable to non-rational transfer functions
- A time delay is described by $G_d(s) = e^{-sT}$
- It does not introduce additional poles or zeros within the contour
- However delay introduces phase lag and consequently has a destabilizing effect.

Effect of Time Delay

$$T = d / v$$

$$L(s) = G(s)G(s)e^{-sT}$$

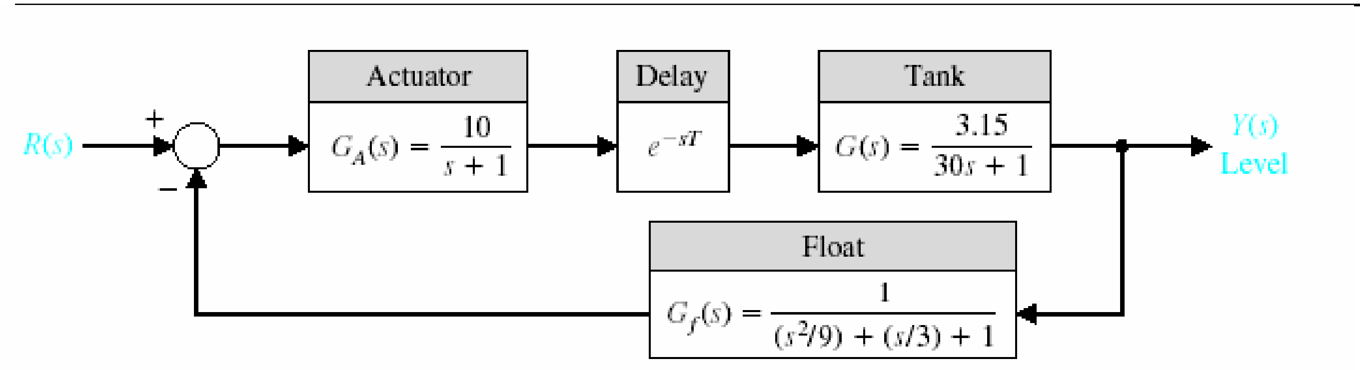


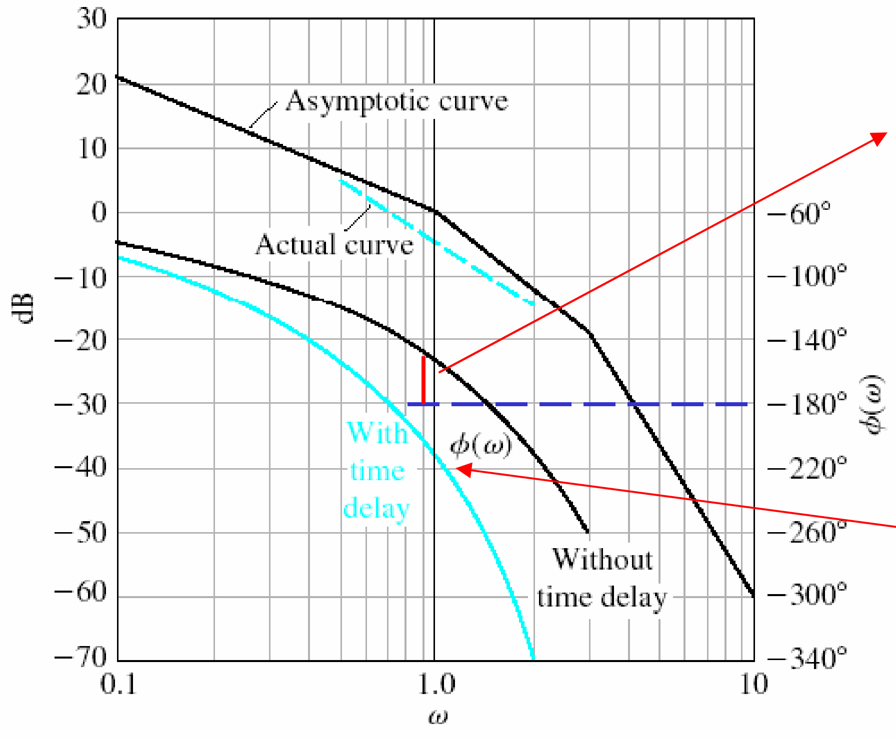
$e^{-j\omega T}$ adds a phase shift $\phi_T = -\omega T$

At the crossover frequency ω_c, $\phi_T = -\omega_c T$

might be sufficient to destabilize the system or
to significantly reduce the phase margin

Effect of Time Delay





Phase margin for delay-free system

Delayed system is unstable!

System with Delay

Consider the following delayed system:

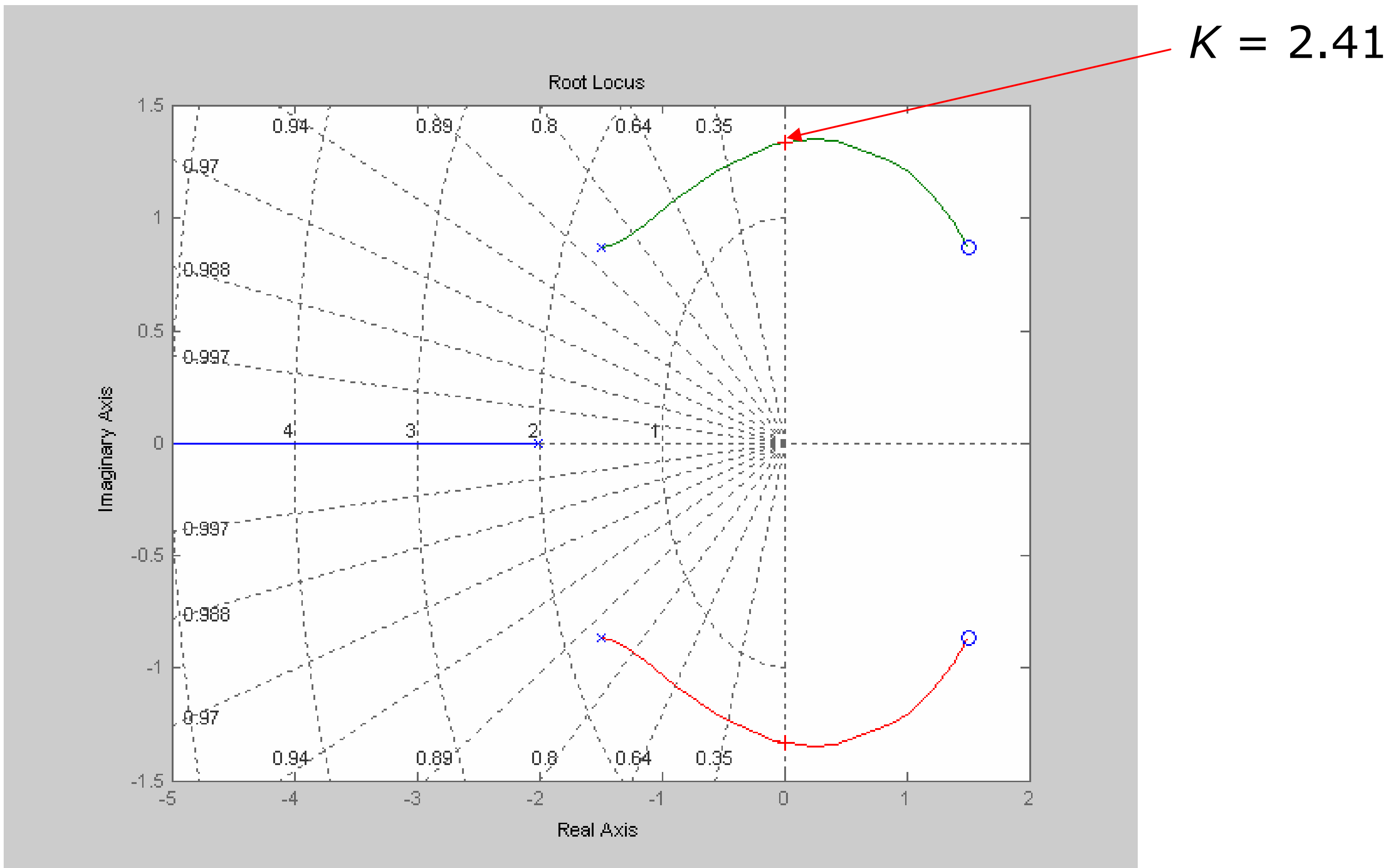
$$G(s) = \frac{e^{-2s}}{s+2}$$

With a second-order Pade approximation, the delay can be written as

$$e^{-2s} \approx \frac{s^2 - 3s + 3}{s^2 + 3s + 3}$$

Hence $$G(s) \approx \frac{s^2 - 3s + 3}{s^3 + 5s^2 + 9s + 6}$$

System with Delay



System with Delay